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Fuzzy σ Algebras of Physics[1]

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Following the idea of Zadeh, the concept of a statistical (or fuzzy) σ algebra is introduced. For two extreme cases of classical and quantum statistical σ algebras the representation theorems are proved. The basic feature distinguishing these two cases is the possibility of producing nontrivial superpositions of pure quantum states, which is absent in the classical case.

1. INTRODUCTION

For ordinary Boolean algebras and σ algebras there exist well-known representation theorems of Stone and of Loomis-Sikorski, respectively (see, e.g., Varadarajan, 1968), which now constitute a part of the classical mathematical background.

The main purpose of the present paper is to prove the representation theorem for the class of statistical σ algebras. To achieve this, the theory of orthomodular σ orthoposets is involved here, and using finally the so-called "fundamental theorem of projective geometry" (cf., Varadarajan, 1968; Maeda, 1970), the representation theorem is established, stating that any statistical σ algebra satisfying the superposition principle (and some additional dimension requirement) can be identified with the σ orthoposet consisting of $\perp$-closed subspaces of some inner product vector space over an involutive division ring. On the other hand, the classical statistical σ algebras are shown to be the ordinary σ algebras of subsets of a given set, as might be expected.

The theory we have developed here is interpreted in terms of familiar concepts of classical and quantum mechanics. The result of the comparison

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of these two theories is somewhat surprising, for the only difference between the classical and quantum theories, when they are compared using the language of the statistical σ-algebra approach, is the existence of pairs of pure states in quantum mechanics which can produce a nonrtivial superposition.

2. AXIOMS FOR A STATISTICAL σ ALGEBRA

The concept of a statistical (or fuzzy) σ algebra is a generalization of that of an ordinary σ algebra of sets, replacing the customary relation "an element x belongs to a subset A" by defining only the probability that x belongs to A (cf. Zadeh, 1965).

To be precise, we define a *statistical σ algebra* as a triple $(P, Q, (\cdot,\cdot))$ consisting of two nonempty sets P and Q, whose members will be called *points* and *objects*, respectively, and a function, $(\cdot,\cdot)$ from $P\times Q$ into the unit real interval $[0, 1]$ [the number (p, a), where $p\in P$, $a\in Q$, will be called, after Zadeh, the *degree* or *grade of the membership of p in a*, and interpreted as the probability that p "belongs to" a], satisfying the Axioms 1-7 below.

Axiom 1. If $(p, a)=(p, b)$ for all p in P, then $a=b$.

Axiom 2. There exists an element a_1 in Q such that $(p, a_1)=1$ for all $p\in P$.

Axiom 3. For every $a\in Q$ there exists $b\in Q$ such that

$$(p, a)+(p, b)=1$$

for all $p\in P$.

Note that by Axiom 1 the elements a_1 in Axiom 2 and b above are unique, and will subsequently be denoted by 1 and a', respectively. The element $1'$ will be denoted by 0. Note also that using the function $(\cdot,\cdot)$ one can define the relations of the *partial ordering* and *orthogonality* in Q following the well-known prescription of Mackey (1963):

$a\leq b$ iff $(p, a)\leq(p, b)$ for all p in P;

$a\perp b$ iff $(p, a)+(p, b)\leq 1$ for every $p\in P$, or, equivalently, iff $a\leq b'$ (or $b\leq a'$.)

Axiom 4. For any sequence $\{a_i\}$ of pairwise orthogonal elements of Q (i.e., satisfying $a_i\perp a_j$, when $i\neq j$) there is an element $a\in Q$ such that

$$(p, a)=\sum_i (p, a_i)$$

for all $p\in P$.

By Axiom 1, the element a above is easily seen to be uniquely determined by the sequence $\{a_i\}$, and it will in the sequel be denoted by $\sum_i a_i$. Note that Axiom 4 simply expresses the fact that Q is σ complete (or, to be more precise, σ-orthocomplete; cf. Mackey, 1963).

Before going further we make a trivial remark that if for some $a, b \in Q$ there exists a point p in P with $(p, a) = 1$ and $(p, b) > 0$, then clearly $a \not\perp b$. The content of our next axiom is that the above implication can be reversed.

Axiom 5. if $a \not\perp b$, where $a, b \in Q$, then there is a point p in P such that $(p, a) = 1$ and $(p, b) > 0$.

The significance of Axion 5 will be clearer if one observes that this axiom can be divided into two parts, whose meaning seems to be simpler than that of Axiom 5, namely:

Axiom 5'. For every nonzero object $a \in Q$ there exists a point $p \in P$ such that $(p, a) = 1$.

Axiom 5''. If for each point $p \in P$ satisfying $(p, a) = 1$ we always have $(p, b) = 1$, where $a, b \in Q$, then $a \leq b$.

To prove the equivalence between Axiom 5 and the pair of Axioms 5', 5'', note first that the implication from Axiom 5 to Axiom 5' is trivial, for it is sufficient to insert $b = 1$ into Axiom 5. (Note that $a \not\perp 1$, 7since a was assumed to be nonzero.) To show that Axiom 5'' also follows as a consequence of Axiom 5, assume the contrary, i.e., that $(p, a) = 1$ implies $(p, b) = 1$ for every p in P, and still $a \not\leq b$. Then $a \not\perp b'$, so that by Axiom 5 there exists a point $p \in P$ with $(p, a) = 1$ and $(p, b') > 0$, the latter implying $(p, b) < 1$, which contradicts our assumption. To prove the converse, i.e., the implication from the pair of Axioms 5', 5'' to Axiom 5, let $a \in Q$, $a \neq 0$, and suppose that $a \not\perp b$ for some b in Q. Then $a \not\leq b'$, so that there must exist by Axion 5'', a point $p \in P$ such that $(p, a) = 1$ and $(p, b') < 1$, the latter being equivalent to $(p, b) > 0$. [Note that the existence of at least one point p with $(p, a) = 1$ is guaranteed by axiom 5'.] The equivalence is therefore established.

In the sequel, we shall write $p \in a$ whenever $(p, a) = 1$ and say that "p belongs to a" or "a contains p." Otherwise, i.e., when $(p, a) < 1$, we write $p \notin a$. Note that we in general do not have the "classical implication" $p \notin a \Rightarrow p \in a'$.

Axiom 6. For each $p \in P$ there is an object $a \in Q$ which contains only the point p. More formally,

$$\forall_{p \in P} \exists_{a \in Q}, \quad p \in a \quad \text{and} \quad \forall_{q \in P, q \neq p}, \quad q \notin a$$

Axiom 7. If $(p, a) > 0$, where $p \in P$ and $a \in Q$, then there exists one and only one point $q \in a$ such that $(p, a) = (p : q)$, where $(p : q)$, the so-called transition probability from p to q, is defined by

$$(p : q) = \inf\{(p, a):\ a \in Q, (q, a) = 1\}.$$

The unique point q, whose existence is guaranteed by Axiom 7, will in the sequel be denoted by p_a. The physical interpretation of p_a is as follows (cf., e.g., Guz, 1981b): p_a is the final pure state of a physical system to which the initial pure state p goes, after the "measurement" of the degree of membership (p, a) of p in a is performed.

To give some examples of mathematical structures satisfying Axioms 1-7, let us first consider any σ algebra $\mathscr{A}$ of subsets of a set X having the property that all one-point subsets of X belong to $\mathscr{A}$. Identifying $P = X$, $Q = \mathscr{A}$, and putting by definition

$$(p, a) = \begin{cases} 1 & \text{if } p \in a \\ 0 & \text{if } \notin a \end{cases}$$

we arrive at a trivial example of a statistical σ algebra $(P, Q, (\cdot, \cdot))$.

A more interesting example can be obtained when one identifies Q with the ortholattice $L(H)$ of the closed subspaces of a complex Hilbert space H, P with the subset of $L(H)$ consisting of the one-dimensional subspaces of H, and puts by definition

$$(p, a) = \operatorname{tr}(P_p P_a)$$

where P_p, P_a denote the orthoprojectors onto p and a, respectively, and tr stands for the trace. In this case, the Axioms 1-7 can be interpreted in terms of pure states and propositions (yes-no measurements). The details of this approach can be found, for instance, in Guz (1981a, b).

3. THE EMBEDDING THEOREM

Now, we shall show the most important consequences of Axioms 1-6, culminating in proving an embedding theorem for Q. The set of objects. The consequences of Axiom 7 will be examined separately at the end of this section.

Theorem 1. The set Q of objects, endowed with the partial ordering $\leq$ and the correspondence $a \mapsto a'$, is a σ orthoposet, i.e., an orthomodular orthocomplemented σ-orthocomplete partially ordered set with the least and the greatest elements, 0 and 1, respectively, in it.

To prove the theorem above we shall adopt the techniques used in the so-called quantum logic approach to axiomatic quantum mechanics (cf.,eg., Mackey, 1963). The proof will be preceded by two lemmas.

Lemma 1. If $a_1 \in Q$, $i = 1, 2, \ldots$, are pairwise orthogonal, then $\sum_i a_i$ is the least upper bound (l.u.b.) for the sequence $\{a_i\}$, denoted subsequently by $\bigvee_i a_i$.

Proof (after Mackey, 1963). By definition, $\sum_j a_i \geq a_i$ for every $i = 1, 2, \ldots$. Suppose now that $b \geq a_i$ for each i. Note that the latter can equivalently be expressed as $b' \perp a_i$ for all i, so that the Axiom 4, when applied to the sequence $\{b', a_1, a_2, \ldots\}$ (consisting, clearly, of pairwise orthogonal elements), leads to the existence of the element $b' + a_1 + a_2 + \cdots$ in Q. In other words, for all $p \in P$ we have

$$\begin{aligned} 1 &\geq (p, b' + a_1 + a_2 + \cdots) \\ &= (p, b') + (p, a_1) + (p, a_2) + \cdots \\ &= (p, b') + \left(p, \sum_j a_j\right) \end{aligned}$$

and hence

$$b' \perp \sum_j a_j$$

or equivalently

$$\sum_j a_j \leq b'' = b$$

The last inequality shows that $\sum_j a_j$ is indeed the l.u.b. for the orthogonal sequence $\{a_j\}$, as claimed. ■

Lemma 2. Q is orthomodular, that is, it has the following property:

$$a \leq b \Rightarrow \exists_{c \in Q,\, c \perp a}, \qquad b = a \vee c$$

Moreover, c is uniquely determined by a and B: $c = b \wedge a'$, the latter denoting the g.l.b. (greatest lower bound) for b and a'.

Proof. Since $a \leq b$, or equivalently $a \perp b'$, we see that by Axiom 4 there exists $a + b'$ in Q.

Letting $c = (a + b')'$, we immediately check out that for an aribtrary $p \in P$,

$$(p, c) = 1 - (p, a + b') = 1 - (p, a) - (p, b') = (p, b) - (p, a)$$

so that

$$(p, b) = (p, a) + (p, c) = (p, a + c) \tag{*}$$

where latter equality is a consequence of the fact that $a \perp c$ (see Axiom 4). Indeed, $c \perp a + b' \geq a$, and hence also $c \perp a$. The equality (*), valid for all

$p \in P$, leads by Axiom 1 to $b = a + c = a \vee c$, where the last equality follows as a consequence of Lemma 1.

Finally, by de Morgan's laws

$$c = (a + b')' = (a \vee b')' = a' \wedge b$$

as claimed. The proof of the lemma is thus complete. ■

We are now in a position to prove our theorem. Clearly, for an arbitrary object $a \in Q$ we have $a'' = a$, and $a \leq b$ implies $b' \leq a'$ $(a, b \in Q)$, which means that the correspondence $a \mapsto a'$ is an involution. We shall show that this involution is in fact an orthocomplementation, i.e., that $b \leq a$ and $b \leq a'$ leads necessarily to $b = 0$. Indeed, if $b \leq a$ and $b \leq a'$ for some $a, b \in Q$, then

$$b \leq \text{g.l.b.}\{a, a'\} = a \wedge a' = (a' \vee a)' = (a' + a)' = 1' = 0$$

so that $b = 0$, as required.

Note that the existence of $a \wedge a' \equiv \text{g.l.b.}\{a, a'\}$ is a direct consequence of de Morgan's law and the existence of the $\text{l.u.b.}\{a', a\} \equiv a' \vee a = a' + a$, the latter being guaranteed by Axiom 4.

The proof of the theorem is therefore complete. Before going further, we need some definition.

An object $a \in Q$ is called a *support* (or *carrier*) of a point $p \in P$ (cf. Zierler, 1961; Pool, 1968), if (i) $p \in a$, i.e., $(p, a) = 1$; (ii) $p \in b$, where $b \in Q$, implies $b \geq a$, i.e., a is the smallest object in Q containing p.

Note that by (ii) the support of p, if it exists, is uniquely determined by the point p. We shall denote it by s_p.

Lemma 3. Every point $p \in P$ has the support, and

$$(q, s_p) < 1$$

for all points $q \neq p$. Moreover, the support s_p of p is an atom in Q, i.e., $a \leq s_p$, $a \in Q$, implies either $a = s$ or $a = 0$, and the correspondence $p \mapsto s_p$, $p \in P$, is a bijection of the set P of points onto the set $A(Q)$ of atmos in Q.

Proof. Let $p \in P$. By Axiom 6, there exists an $a \in Q$ such that $(p, a) = 1$ and (q, a) 1 for all points q different from p. We shall show that a is the support of p.

Let $b \in Q$, $b \not\geq a$. (Note that such an element b always exists, since $a \neq 0$, so one can take, for instance, $b = 0$). Since $a \not\perp b'$ and $a \neq 0$, there exists by Axiom 5 a point $r \in P$ such that

$$(r, a) = 1 \qquad \text{and} \qquad (r, b') > 0$$

the latter being equivalent to $(r, b) < 1$. However, by virtue of Axiom 6, the point r must be identical with p, $r = p$, and we therefore conclude that

$(p, b) < 1$. Thus, we have shown that

$$\forall_{p \in P} \exists_{a \in Q} \Big((p, a) = 1 \text{ and } \forall_{b \in Q,\, B \not\geq a} (p, b) < 1 \Big)$$

which means that the object a is the support of p, $a = s_p$.

At the same time we have proved (see above) that $(q, a) < 1$ for every point $q \neq p$.

Now, we shall show that s_p is an atom in Q.

Suppose that $c \leq s_p$, $c \neq 0$. By Axiom 5′, there exists a point $q \in P$ with $(q, c) = 1$, and hence also $(q, s_p) = 1$, so that $q = p$ according to what we have already proved. But $(q, c) = 1$ implies $c \geq s_q = s_p$, and therefore $c = s_p$, as claimed.

To prove that $p \mapsto s_p$ is one-one, assume that $s_p = s_q$ for some $p, q \in P$. Then

$$(q, s_p) = (q, s_q) = 1$$

and hence $q = p$.

Finally, if e is an atom in Q, then by Axiom 5′ we have

$$(p, e) = 1$$

for some $p \in P$, and hence $e \geq s_p$, so that $e = s_p$, as e is already an atom. This proves that $p \mapsto s_p$ is "onto," and concludes the proof of the lemma. ■

Lemma 4. Q is atomic, that is for every nonzero element $a \in Q$ there exists an atom $e \in Q$ such that $e \leq a$. Moreover, Q is atomistic, i.e., every nonzero element $a \in Q$ is the l.u.b. of the atoms contained in it. More precisely,

$$a = \bigvee \{ s_p \colon p \in P,\ (p, a) = 1 \}$$

Proof. Let a be a nonzero element of Q. By Axiom 5′, there exists a point $p \in P$ with $(p, a) = 1$. Hence $a \geq s_p$, where s_p is an atom by Lemma 3, which concludes the proof of the first half of the lemma.

Obviously, $a \geq s_p$ for every $p \in P$ with $(p, a) = 1$. Now, assume that for some $b \in Q$ we also have $b \geq s_p$ for all $p \in P$ satisfying $(p, a) = 1$, and prove that $b \geq a$. Since $b \geq s_p$, we have $(p, b) \geq (p, s_p) = 1$, so that $(p, b) = 1$. The latter equality, valid for all $p \in P$ with $(p, a) = 1$, shows that we must have $a \leq b$ indeed, after we take into account our Axiom 5″.

To summarize the results we have obtained so far as the consequences of Axioms 1–6, we state the following theorem.

Theorem 2. Let $(P, Q, (\cdot, \cdot))$ be a statistical σ algebra. Then Q is an atomistic σ orthoposet, and there is a one-to-one mapping $p \mapsto s_p$ of the set P of points onto the set $A(Q)$ of atoms in Q such that $s_p \leq a$ if and only if $(p, a) = 1$.

Before going further, we have to introduce some definitions. We shall say that two points $p, q \in P$ are *orthogonal* (compare Gudder, 1970), and write $p \perp q$, if for some object $a \in Q$ we have $(p, a) = 1$ and $(q, a) = 0$. The above-defined orthogonality relation $\perp$ is clearly symmetric. Note, by the way, that $p \perp q$ if and only if $s_p \perp s_q$. Indeed, $p \perp q$ implies $s_p \leq a$ and $s_q \leq a'$ for some $a \in Q$, and hence $s_p \perp s_q$. Conversely, $s_p \perp s_q$ leads to

$$1 = (p, s_p) \leq (p, s_q')$$

so that $(p, s_q') = 1 = (q, s_q)$, which shows that $p \perp q$, as claimed.

The set P of all points endowed with the orthogonality relation defined above will be called the *generalized phase space.* Thus, the generalized phase space $(P, \perp)$ provides an example of what is called by mathematicians an *orthogonality space* (cf., e.g., Randall et al., 1972; Gerelle et al., 1974, and references quoted therein).

Let now S be a subset of P. Define S to be the set of all points $p \in P$ such that $p \perp S$ (read: $p \perp q$ for all $q \in S$), and write S^- instead of $S^{\perp\perp}$. Clearly, $s \subseteq S^-$, and when $S = S^-$, we call the set S *closed* (or, to be more precise, $\perp$ *closed*). It is not difficult to check out that under set inclusion the family of all closed subsets of P, denoted subsequently by $C(P, \perp)$ and called the *phase geometry* associated with $(P, Q, (\cdot, \cdot))$ (Guz, 1975) becomes a complete orthocomplemented lattice, whose lattice operations (joins and meets, respectively) are given by

$$\bigvee_j S_j = (\bigcup_j S_j)^-, \qquad \bigwedge_j S_j = \bigcap_j S_j$$

where $\{S_j\}$ denotes an arbitrary family of closed subsets of P, and the orthocomplementation is given by the correspondence $S \mapsto S^\perp$, $S \in C(P, \perp)$. For the empty set $\varnothing$ we put, by definition, $\varnothing^\perp = P$, so that both $\varnothing$ and P belong to $C(P, \perp)$.

Lemma 5. Let S be a nonempty subset of P. Then

$$S^- = \{p \in P\colon (p, a) = 1 \text{ for all } a \in Q \text{ such that } (q, a) = 1 \text{ for every } q \in S\}$$

Proof. Let $p \in P$ be such that $(p, a) = 1$ for every $a \in Q$ satisfying $(q, a) = 1$ for all $q \in S$. We shall show that $p \in S^-$, i.e., that $p \perp S^\perp$. To do this, two cases should be considered.

(a) $S^\perp = \varnothing$. In this case, $S^- = \varnothing^\perp = P$, so nothing has to be proved.

(b) $S^\perp \neq \varnothing$. Let $r \in S$. Then $s_r \perp s_q$ for every $q \in S$, so that $(q, s_r') = 1$ for all $q \in S$. By the hypothesis, $(p, s_r') = 1$, so that $s_p \leq s_r'$ or $s_p \perp s_r$, that is, $p \perp r$. We thus have shown that $p \perp r$ for every $r \in S^\perp$, which means that $p \perp S^\perp$ or $p \in (S^\perp)^\perp = S^-$.

To prove the opposite inclusion, assume that $p \in S^-$, i.e., $p \perp S^\perp$. We shall show that $(p, a) = 1$ for every $a \in Q$ satisfying $(q, a) = 1$ for all $q \in S$.

Again, two cases will be considered. One can assume, without any loss of generality, that $a \neq 1$. Then, by Lemma 5,

$$a' = \bigvee\{s_r\colon r \in P,\ (r, a') = 1\}$$

Since for an arbitrary $q \in S$, $s_r \leq a' \leq s'_q$, or, equivalently, $r \in S^\perp$, one must in particular have, by the assumption $p \perp r$, so that $s_p \perp s_r$. Hence also

$$s_p \perp \bigvee\{s_r\colon r \in P,\ (r, a') = 1\} = a'$$

and therefore $(p, a) \geq (p, s_p) = 1$, so that $(p, a) = 1$, as claimed. The proof of the lemma is therefore complete. ■

The physical interpretation of the members of the set difference $S^-\backslash S$ is, according to Varadarajan (1968), that $S^-\backslash S$ represent the set of *superpositions* of the pure states from S, as these elements are precisely the "pure states" which have all the properties possessed by all the elements of S simultaneously (for details, see Varadarajan, 1968).

We shall now prove the following "embedding theorem."

Theorem 3. For every $a \in Q$, the set $a^1 = \{p \in P\colon (p, a) = 1\}$ belongs to $C(P, \perp)$, and the correspondence $a \mapsto a^1$ defines an orthoinjection of the σ-orthoposet Q into $C(P, \perp)$.

Proof. Let us first observe that

$$a^1 = (a^0)^\perp \qquad (**)$$

where

$$a^0 = \{q \in P\colon (q, a) = 0\}$$

hence

$$(a^1)^- = (a^0)^{\perp\perp\perp} = (a^0)^\perp = a^1$$

which shows that $a^1 \in C(P, \perp)$.

To prove (**), one can assume with no loss of generality that $a \neq 0$, since $a = 0$ implies $a^0 = P$, so that $(a^0)^\perp = P^\perp = \varnothing = a^1$, as required.

Let $p \in a^1$. Then clearly, $p \perp q$ for all $q \in a^0$, that is, $p \in (a^0)^\perp$, which proves that $a^1 \subseteq (a^0)^\perp$. To prove the opposite inclusion, assume that $p \in (a^0)^\perp$, so that $s_p \perp s_q$ for all $q \in a^0$. Hence $s_p \perp a'$, since by Lemma 4, $a' = \bigvee\{s_q\colon q \in P,\ q \in a^0\}$. Therefore,

$$(p, a) \geq (p, s_p) = 1$$

so that $(p, a) = 1$, which shows that $p \in a^1$, and the opposite inclusion is also established.

To prove the second half of the theorem, observe that $a \leq b$, where $a, b \in Q$, implies clearly $a^1 \subseteq b^1$, and that the opposite implication is guaranteed by Axiom 5″. Obviously, $0 = {}^1\varnothing$, and $1^1 = P$, so there remains to be shown that the correspondence $a \mapsto a^1$ preserves the orthocomplementation, i.e.,

$$a'^1 = (a^1)$$

But $a^1 = a'^0$, and one therefore finds by using (**) that

$$(a^1) = (a'^0) = a'^1$$

as required. The proof of the theorem is thus complete. ■

We shall now come back to Axiom 7 and its consequences. The physical significance of this axiom has been clarified in a series of papers (Bugajska and Bugajski, 1973a, b; Guz, 1980, 1981a, b), where several equivalent forms of this postulate were found and analysed in detail. In particular, the equivalence between Axiom 7 and the so-called covering law in Q has been established. (We recall that the *covering law* holds in Q, or that Q possesses the *covering property*, if for each $a \in Q$ and each atom $e \in A(Q)$ there exists $a \vee e$ in Q, and $a \vee e$ covers a, when $e \not\leq a$, i.e., $a \vee e \geq b \geq a$ implies either $b = a$ or $b = a \vee e$). The latter is in turn equivalent to the well-known *Jauch-Piron condition* in Q (see Jauch and Piron, 1969), Stating that for every pair a, e of elements of Q, where e is an atom, the "difference" $a \vee e - a$, defined as $(a \vee e) \wedge a'$, is either an atom (when $e \not\leq a$) or zero (when $e \leq a$).

So, we finally arrive at the following result.

Theorem 4. If $(P, Q, (\cdot,\cdot))$ is a statistical σ algebra, then Q, the set of objects, endowed with the partial ordering $\leq$ and the orthocomplementation′, is an atomistic σ orthoposet satisfying the covering law.

Moreover, repeating the arguments used previously in the context of the quantum logic approach (Guz, 1978) one can establish the following fact.

Theorem 5. For any statistical σ algebra $(P, Q, (\cdot,\cdot))$, its associated phase geometry $C(p, \perp)$ is an atomistic, orthomodular, orthocomplemented complete lattice with the covering law holding in it.

4. SUPERPOSITION PRINCIPLE

A pair $\{p, q\}$ consisting of two distinct points from P is said to be *classical* if $\{p, q\}^- = \{p, q\}$, that is, if there is no superposition of p and q. Otherwise, it is called *nonclassical.*

Now, we have two extreme possibilities for statistical σ algebras: the *quantum* case, distinguished by the validity of the *superposition principle* (cf. Guz, 1974, 1975; Pulmannova, 1976) which says that any pair $\{p, q\}$ is nonclassical, i.e., there exists a third point (pure state) in P, different from both p and q, which is a superposition of p and q; and the *classical* case, where there are no nonclassical pairs at all. The corresponding statistical σ algebras will be called *quantum* and *classical*, respectively.

It is not difficult to show that in the classical case the transition probability function (:) is trivial, that is,

$$(p{:}q)=\begin{cases}1 & \text{if } p=q\\ 0 & \text{if } p\neq q\end{cases}$$

Indeed, suppose that (p, q) is a classical pair, i.e., $\{p, q\}^{-}=\{p, q\}, p\neq q$. Then, passing on to the supports s_p, s_q of, respectively, p and q, we find by using the orthomodularity of Q

$$s_p\vee s_q=(s_p\vee s_q-s_q)+s_q$$

where $s_p\vee s_q-s_q$ is again an atom by the Jauch–Picron condition. Thus, by Lemma 3, $s_p\vee s_q-s_q=s_r$ for some $r\in P$. Obviously $r\in\{p, q\}^{-}$. Indeed, assuming $(p, a)=(q, a)=1$ for some $a\in Q$, one immediately finds that $s_q\vee s_q\leq a$, and therefore also $s_r\leq a$, which means that $(r, a)=1$, the latter showing that $r\in\{p, q\}^{-}$, as claimed. But, by the assumption, r must equal either p or q. Since $s_r\perp s_q$, we find $r=p$ and, clearly, $p\perp q$. Hence, $(p{:}q)=(p, s_q)\leq(p, s_p')=0$, so that $(p:p)=0$. Thus, in the classical case, we have $(p:q)=0$ for all pairs (p, q) with $p\neq q$ (obviously, $(p:p)=1$ for every $p\in P$), which proves our statement.

Moreover, in the classical case one has $p_1\in a^1$ if and only if $(p, a)=0$, since $p\notin a^1$ implies now $p\perp a^1$, and hence $s_p\perp\vee\{s_q\colon q\in a^1\}$, the latter equality being valid by Lemma 4; hence $(p, a)\leq(p, s_p')=0$, so that $(p, a)=0$. Thus, in the classical case the function $(\cdot,\cdot)$ is of the form

$$(p, a)=\begin{cases}1 & \text{if } p\notin a^1\\ 0 & \text{if } p\notin a^{1\cdot}\end{cases}$$

so, we have now arrived at the trivial "degree of membership" function discussed previously in Section 2.

We thus have shown that for any classical statistical σ algebra $(P, Q, (\cdot,\cdot))$, in which all two-point subsets $\{p, q\}\subseteq P$ are classical pairs, the set Q of objects can be identified (by Theorem 3) with a σ algebra of subsets of P, and the function $(\cdot,\cdot)$ is degenerated in the sense that $(p,\cdot)$ is the δ measure on Q concentrated at the point p.

In the quantum case, the validity of the superposition principle leads to the irreducibility of the phase geometry $C(P,\perp)$ (Guz, 1978), which together with Theorem 5 enables us to apply in this case the so-called "fundamental theorem of projective geometry" (cf., e.g., Varadarajan, 1968; Maeda, 1970), provided we add to Axioms 1–7 the requirement that the projective dimension of $C(P, \perp)$ is greater than 3 (which simply means that there are at least four orthogonal points in P). Then, by the above-mentioned theorem, there exists an inner product vector space $(V, \langle\cdot,\cdot\rangle)$ over an involutive division ring D such that $C(P, \perp)$ is orthoisomorphic to the lattice of $\perp$-closed subspaces of the vector space V. (We recall that a subspace M of V is said to be *$\perp$-closed* if $M = M^{\perp\perp}$, where $M^{\perp} = \{x \in V: \langle x, y\rangle = 0\}$ for all $y \in M$). Clearly, the representation theorem above applies also to Q, as the latter was identified by Theorem 3 as a sub-σ-orthoposet of $C(P, \perp)$.

Summarizing the results obtained so far, we arrive at the following representation theorem.

Theorem 6. For any classical statistical σ algebra $(P, Q, (\cdot,\cdot))$, the set Q of objects can be identified with some σ algebra of subsets of the set P, and the function $(\cdot,\cdot)$ becomes then trivial, i.e., for every p, $(p,\cdot)$ is the δ measure concentrated at p. If $(P, Q, (\cdot,\cdot))$ is a quantum statistical σ algebra such that there exist at least four orthogonal points in P, then there is as inner product vector space $(V, \langle\cdot,\cdot\rangle)$ over an involutive division ring D such that Q can be identified with a σ orthoposet consisting of $\perp$-closed subspaces of the vector space V.

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